Reconstruction of higher stage first class constraints into the secondary ones

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Abstract

Starting from a Lagrangian theory L with first class constraints up to N-th stage, we construct an equivalent Lagrangian $\tilde{L}$ with at most secondary first class constraints presented in a Hamiltonian formulation. The Lagrangian $\tilde{L}$ can be obtained by pure algebraic methods, it's manifest form in terms of quantities of the initial formulation is find. Local symmetries of $\tilde{L}$ are find in closed form. All the constraints of L turns out to be gauge symmetry generators for $\tilde{L}$.

1 Introduction

Search for constructive and simple method of finding all the local symmetries of a given Lagrangian action is an interesting problem under investigation [1-5]. Analysis of the problem is usually carried out in a Hamiltonian formulation [6, 7]. For a theory with first class constraints only, local symmetries of a complete Hamiltonian action can be determined [2] starting from an extended Hamiltonian action[1], for the latter a complete and irreducible set of symmetries is known in closed form [8]. While it gives relatively simple algorithm, some of it's points remain unclarified. In particular, the completeness and irreducibility of the resulting set were not demonstrated

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[1]By definition, the extended Hamiltonian is obtained from the complete one by addition of higher-stage (secondary, tertiary, ...) constraints with the corresponding Lagrangian multipliers. It is known [7] that the two formulations are equivalent. It is interesting to note that the equivalence is not essential for the algorithm suggested in [2].

so far [3]. Analysis of a general case (first and second class constraints are presented) turns out to be much more complicated issue [3-5]. So, it may be reasonable to adopt a different approach to the problem. Namely, instead of looking for symmetries of the initial Lagrangian, one can search for an equivalent Lagrangian formulation that admits a simple solution of the problem. Below, we test such a kind possibility for the case of a theory with first class constraints[2]. Besides, we clarify a relation among formulations of a constrained system in terms of the complete and the extended Hamiltonians.

The work is organized as follows. With the aim to fix our notations, we describe in Section 2 the Hamiltonization procedure for a Lagrangian theory with first class constraints up to N-th stage presented. In Section 3 we formulate pure algebraic recipe for reconstruction of original Lagrangian into the one that admits at most secondary first class constraints. The corresponding complete Hamiltonian turns out to be closely related with the extended Hamiltonian of original formulation (modulo to trivial constraints, they coincide after identification of some configuration space variables of the reconstructed formulation with the Lagrangian multipliers for higher stage constraints of the original formulation, see Eq. (10) below). Since the original and the reconstructed formulations are equivalent, it is matter of convenience to use one or another of them for description of a theory under investigation[3]. In Section 4 we demonstrate one of advantages of the reconstructed formulation by finding it's complete irreducible set of local symmetries. The procedure is illustrated on example of a model with fourth-stage constraints presented.

2 Initial formulation with higher stage constraints

Let $L(q^A, \dot{q}^B)$ be Lagrangian of singular theory: $rank\frac{\partial^2 L}{\partial\dot{q}^A\partial\dot{q}^B} = [i] < [A]$, defined on configuration space $q^A, A = 1, 2, \ldots, [A]$. From the beginning, it is convenient to rearrange the initial variables in such a way that the rank minor is placed in the upper left corner. Then one has $q^A = (q^i, q^\alpha)$, $i = 1, 2, \ldots, [i]$, $\quad \alpha = 1, 2, \ldots, [\alpha] = [A] - [i]$,

[2]Second class constraints can be converted into the first class constraints of higher stages, see [9].

[3]Let us point out also that higher stage constraints usually appear in a covariant form. So one expects manifest covariance of the reconstructed formulation.

where $\det \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \neq 0$.

Let us construct Hamiltonian formulation for the theory. To fix our notations, we carry out the Hamiltonization procedure in some details. One introduces conjugate momenta according to the equations $p_A = \frac{\partial L}{\partial \dot{q}^A}$, or

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{1}$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}. \tag{2}$$

They are considered as algebraic equations for determining of velocities $\dot{q}^A$. According to the rank condition, $[i]$ equations (1) can be resolved with respect to $\dot{q}^i$, let us denote the solution as

$$\dot{q}^i = v^i(q^A, p_j, \dot{q}^\alpha). \tag{3}$$

It can be substituted into remaining $[\alpha]$ equations for the momenta (2). By construction, the resulting expressions do not depend on $\dot{q}^A$ and are called primary constraints $\Phi_\alpha(q,p)$ of the Hamiltonian formulation. One finds

$$\Phi_\alpha \equiv p_\alpha - f_\alpha(q^A, p_j) = 0, \tag{4}$$

where

$$f_\alpha(q^A, p_j) \equiv \left. \frac{\partial L}{\partial \dot{q}^\alpha} \right|_{\dot{q}^i = v^i(q^A, p_j, \dot{q}^\alpha)}. \tag{5}$$

The equations (1) (2) are thus equivalent to the system (3), (4).

Next step of the Hamiltonian procedure is to introduce an extended phase space parameterized by the coordinates q^A, p_A, v^α, and to define a complete Hamiltonian H according to the rule

$$H(q^A, p_A, v^\alpha) = H_0(q^A, p_j) + v^\alpha \Phi_\alpha(q^A, p_B), \tag{6}$$

where

$$H_0 = \left. (p_i \dot{q}^i - L + \dot{q}^\alpha \frac{\partial L}{\partial \dot{q}^\alpha}) \right|_{\dot{q}^i \to v^i(q^A, p_j, \dot{q}^\alpha)}. \tag{7}$$

Then the following system of equations on this space

$$\dot{q}^A = \{q^A, H\}, \qquad \dot{p}_A = \{p_A, H\}, \qquad \Phi_\alpha(q^A, p_B) = 0, \tag{8}$$

is equivalent to the Lagrangian equations following from L, see [7]. Here $\{,\}$ denotes the Poisson bracket. Let us point that equations for $\dot{q}^i$ of the system (8) coincide, modulo to notations, with Eq. (3), where $\dot{q}^\alpha$ are replaced by v^α (see [10] for more details). This fact will be used in the next section.

It may happens, that the system (8) contains in reality more then $[\alpha]$ algebraic equations. Actually, derivative of the primary constraints with respect to time implies the so called second stage equations as algebraic consequences of the system (8): $\{\Phi_\alpha, H\} = 0$. Let us suppose that on-shell these expressions do not involve the Lagrangian multipliers v^α. Functionally independent equations of the system, if any, represent then secondary Dirac constraints $\Phi^{(2)}_{\alpha_2}(q^A, p_j) = 0$. They may imply third-stage constraints, and so on. We suppose that the theory has constraints up to N-th stage, $N > 2$. Higher stage constraints (that is those of second stage, third stage, ...) are denoted by $T_a(q^A, p_j) = 0$. Then the complete constraint system is $G_I \equiv (\Phi_\alpha, , T_a)$. In this work we restrict ourselves to the case of a theory with first class constraints only

$$\{G_I, G_J\} = c_{IJ}{}^K(q^A, p_j)G_K, \qquad \{G_I, H_0\} = b_I{}^J(q^A, p_j)G_J, \tag{9}$$

where c, b are phase space functions. Since the quantities on l.h.s. of these equations are at most linear on p_α, one has: $c_{IJ}{}^\alpha = 0$, $b_I{}^\alpha = 0$.

3 Reconstruction of higher stage constraints into at most secondary ones

In this section we realize the following program. Starting from the theory described above, we construct on configuration space q^A, s^a the Lagrangian $\tilde{L}(q^A, \dot{q}^A, s^a)$, which leads to the Hamiltonian[4] $H_0 + s^a T_a$, and to the primary constraints $\Phi_\alpha = 0$, $\pi_a = 0$, where π_a represent conjugate momenta for s^a. Due to special form of the Hamiltonian, preservation in time of the primary constraints implies, that all the higher stage constraints of initial theory appear as

[4] Let us stress once again, that in our formulation the variables s^a represent a part of the configuration-space variables.

secondary constraints for the theory $\tilde{L}$. The Dirac procedure stops on the second stage. Besides, we demonstrate that the formulations L and $\tilde{L}$ are equivalent.

We start with the following auxiliary construction. On a space parameterized by the coordinates $q^A, \tilde{p}_A, s^a, \pi_a, v^\alpha, v^a$, let's consider the following function[5]

$$\tilde{H}(q^A, \tilde{p}_A, s^a, \pi_a, v^\alpha, v^a) = \tilde{H}_0(q^A, \tilde{p}_j, s^a,) + v^\alpha \Phi_\alpha + v^a \pi_a, \qquad (10)$$

where

$$\tilde{H}_0 = H_0 + s^a T_a, \qquad (11)$$

and the functions $H_0(q^A, \tilde{p}_j), T_a(q^A, \tilde{p}_j)$ are taken from the initial formulation. One writes the equations

$$\dot{q}^i = \frac{\partial \tilde{H}}{\partial \tilde{p}_i} = \frac{\partial H_0}{\partial \tilde{p}_i} + s^a \frac{\partial T_a}{\partial \tilde{p}_i} - v^\alpha \frac{\partial f_\alpha}{\partial \tilde{p}_i}. \qquad (12)$$

They can be resolved algebraically with respect to $\tilde{p}_i$ in a neighboard of the point $s^a = 0$. Actually, Eq. (12) with $s^a = 0$ is $\dot{q}^i = \frac{\partial H}{\partial \tilde{p}_i}$, that is the equation $\dot{q}^i = v^i(q^A, p_j, \dot{q}^\alpha)$ of the initial theory. It's solution exists, and is written in Eq. (1). Hence det $\frac{\partial^2 \tilde{H}}{\partial \tilde{p}_i \partial \tilde{p}_j} \neq 0$ at the point $s^a = 0$. Then the same is true in some vicinity of this point, and Eq. (12) thus can be resolved. Let us denote the solution as

$$\tilde{p}_i = \omega_i(q^A, \dot{q}^i, v^\alpha, s^a). \qquad (13)$$

By construction, one has the identities

$$\omega_i\Big|_{\dot{q}^i = \frac{\partial \tilde{H}}{\partial \tilde{p}_i}} \equiv \tilde{p}_i, \qquad \left.\frac{\partial \tilde{H}}{\partial \tilde{p}_i}\right|_{\tilde{p}_i = \omega_i} \equiv \dot{q}^i, \qquad (14)$$

as well as the following property of ω

$$\omega_i(q^A, \dot{q}^i, v^\alpha, s^a)\Big|_{s^a = 0, v^\alpha \to \dot{q}^\alpha} = \frac{\partial L}{\partial \dot{q}^i}. \qquad (15)$$

Below we use the notation

$$\omega_i(q^A, \dot{q}^i, v^\alpha, s^a)\Big|_{v^\alpha \to \dot{q}^\alpha} \equiv \omega_i(q, \dot{q}, s), \qquad (16)$$

[5] Last term in this expression is not necessary for construction of $\tilde{L}$ and can be omitted. It is convenient to keep them, then Eq. (10) turns out to be complete Hamiltonian for $\tilde{L}$ of Eq. (17).

Now, on configuration space parameterized by q^A, s^a, let us define the following Lagrangian[6]

$$\tilde{L}(q^A, \dot{q}^A, s^a) = \left(\omega_i \dot{q}^i + f_\alpha(q^A, \omega_j)\dot{q}^\alpha - H_0(q^A, \omega_j) - s^a T_a(q^A, \omega_j)\right)\Big|_{v^\alpha \to \dot{q}^\alpha}. \quad (17)$$

As compare with the initial Lagrangian, $\tilde{L}$ involves the new variables s^a, in a number equal to the number of higher stage constraints T_a. Considering $\tilde{L}$ as a function of ω, one finds $\frac{\partial \tilde{L}}{\partial \omega_i}\Big|_{\omega(q,\dot{q},s)}$ $= \left(\dot{q}^i - \frac{\partial \tilde{H}}{\partial \tilde{p}_i}\right)\Big|_{\tilde{p}=\omega(q,\dot{q},s)} = 0$, according to the identity (14). Thus the new Lagrangian obeys to the property

$$\frac{\partial \tilde{L}}{\partial \omega_i}\Big|_{\omega(q,\dot{q},s)} = 0, \quad (18)$$

the latter will be crucial for discussion of local symmetries in the next section.

By using of Eq. (7), $\tilde{L}$ can be written also in terms of the initial Lagrangian

$$\tilde{L}(q^A, \dot{q}^A, s^a) = L(q^A, v^i(q^A, \omega_j, \dot{q}^\alpha), \dot{q}^\alpha) + \omega_i(\dot{q}^i - v^i(q^A, \omega_j, \dot{q}^\alpha)) - s^a T_a(q^A, \omega_i), \quad (19)$$

where the functions v^i, ω_i are given by Eqs. (3), (16).

Following to the standard prescription [6, 7], let us construct Hamiltonian formulation for $\tilde{L}$. By using of Eq. (17), one finds conjugate momenta for q^A, s^a

$$\tilde{p}_i = \frac{\partial \tilde{L}}{\partial \dot{q}^i} = \omega_i(q^A, \dot{q}^A, s^a), \qquad \tilde{p}_\alpha = \frac{\partial \tilde{L}}{\partial \dot{q}^\alpha} = f_\alpha(q^A, \omega_j),$$
$$\pi_a = \frac{\partial \tilde{L}}{\partial \dot{s}^a} = 0. \quad (20)$$

Due to the identities (14), these expressions can be rewritten in equivalent form

$$\dot{q}^i = \frac{\partial \tilde{H}}{\partial \tilde{p}_i}, \qquad \tilde{p}_\alpha = f_\alpha(q^A, \tilde{p}_j), \qquad \pi_a = 0, \quad (21)$$

[6]It can be considered as constructed according to the rule $\tilde{L} = \tilde{p}_A \dot{q}^A + \pi_a s^a - \tilde{H}$, where $\tilde{p}_\alpha$ are excluded by using of the constraints (2) related with the initial Hamiltonian H, while $\tilde{p}_j$ are excluded with help of Eq. (13) related with the new Hamiltonian $\tilde{H}$.

Thus the velocities $\dot{q}^i$ have been determined, while as the primary constraints there appear $\pi_a = 0$, and the primary constraints $\Phi_\alpha = 0$ of the initial theory. One finds the Hamiltonian $\tilde{H}_0$

$$\tilde{H}_0 = \tilde{p}_A\dot{q}^A + \pi_a\dot{s}^a - \tilde{L} = H_0 + s^a T_a(q^A, \tilde{p}_j), \tag{22}$$

so the complete Hamiltonian $\tilde{H}$ is given by Eq. (10). Further, preservation in time of the primary constraints $\pi_a = 0$ implies the equations $T_a = 0$. Hence all the higher stage constraints of initial formulation appear now as secondary constraints. Preservation in time of the primary constraints Φ_α leads to the equations $\{\Phi_\alpha, \tilde{H}\} = \{\Phi_\alpha, H\} + s^a c_{\alpha a}{}^I G_I \approx \{\Phi_\alpha, H\} = 0$, that is to the second stage equations of initial formulation. Henceforce, as before, they imply the secondary constraints $\Phi^{(2)}_{\alpha_2} = 0$, the latter appeared already as a part of the set $T_a = 0$. The Dirac procedure stops at the second stage. Owing to the structure of gauge algebra (9), there are no neither higher stage constraints nor equations for determining of the multipliers v^α, v^a.

Let us compare the theories $\tilde{L}$ and L. Dynamics of the theory $\tilde{L}$ is governed by the Hamiltonian equations

$$\dot{q}^A = \{q^A, H\} + s^a\{q^A, T_a\}, \qquad \dot{\tilde{p}}_A = \{\tilde{p}_A, H\} + s^a\{\tilde{p}_A, T_a\},$$
$$\dot{s}^a = v^a, \qquad \dot{\pi}_a = 0, \tag{23}$$

as well as by the constraints

$$\pi_a = 0, \qquad T_a = 0. \tag{24}$$

Here H is complete Hamiltonian of the initial theory (6), and the Poisson bracket is defined on phase space q^A, s^a, p_A, π_a. Let us make partial fixation of a gauge by imposing the equations $s^a = 0$ as a gauge conditions for the constraints $\pi_a = 0$. Then (s^a, π_a)-sector of the theory disappears, while the remaining equations in (23), (24) coincide with those of the initial theory[7] L. Let us remind that $\tilde{L}$ has been constructed in some vicinity of the point $s^a = 0$. Admissibility of the gauge $s^a = 0$ then guarantees a self consistency of the construction. Thus L represents one of the gauges for $\tilde{L}$, which proves equivalence of the two formulations.

[7] In more rigorous treatment, one writes Dirac brackets corresponding to the equations $\pi_a = 0, s^a = 0$. Then the latter can be used before computation of the brackets, that is the variables s^a, π_a can be omitted. For the remaining phase-space variables q^A, p_A, the Dirac bracket coincides with the Poisson one.

4 Restoration of Lagrangian local symmetries from known Hamiltonian constraints

Since the initial Lagrangian is one of gauges for $\tilde{L}$, physical system under consideration can be equally analyzed by using of the reconstructed Lagrangian. Here we demonstrate one of advantages of this description, namely, search for local symmetries of $\tilde{L}$ turns out to be much more easy problem as compare with those problem for L. The reconstructed Lagrangian $\tilde{L}$ implies $[\alpha]+[a]$ primary first class constraints, so one expects the same number of independent local symmetries in the formulation. The symmetries can be easily find in explicit form in terms of the constraints G_I of initial formulation.

Symmetries of Hamiltonian action. We start from discussion of local symmetries for Hamiltonian action[8] which corresponds to $\tilde{L}$

$$
\begin{aligned}
S_{\tilde{H}\tilde{L}} = \int d\tau(\tilde{p}_A\dot{q}^A + \pi_a\dot{s}^a - \tilde{H}) = \\
\int d\tau(\tilde{p}_A\dot{q}^A + \pi_a\dot{s}^a - H_0(q^A,\tilde{p}_j) - s^aT_a(q^A,\tilde{p}_j) - \\
v^\alpha\Phi_\alpha(q^A,\tilde{p}_B) - v^a\pi_a).
\end{aligned} \tag{25}
$$

Let us consider variation of $S_{\tilde{H}\tilde{L}}$ under infinitesimal transformation $\delta_I q^A = \epsilon^I\{q^A, G_I\}$, $\delta_I\tilde{p}_A = \epsilon^I\{\tilde{p}_A, G_I\}$, where ϵ^I are the parameters, and I stands for any fixed α or a. It implies (modulo to total derivative terms which we omit in subsequent computations) $\delta(\tilde{p}_A\dot{q}^A) = \dot{\epsilon}^I G_I$, and $\delta A(q,p) = \epsilon^I\{A, G_I\}$ for any function $A(q,p)$. Owing to these relations, variation of $S_{\tilde{H}\tilde{L}}$ is proportional to Φ_α, T_a, henceforce it can be canceled by appropriate transformation of v^α, s^a. In turn, transformation of s^a implies $\delta(\pi_a\dot{s}^a) = \pi_a(\delta s^a)^\cdot$, which can be canceled by variation of v^a: $\delta v^a = (\delta s^a)^\cdot$. Direct computations show, that the following transformations[9]:

$$
\begin{aligned}
&\delta_I q^A = \epsilon^I\{q^A, G_I\}, \qquad \delta_I\tilde{p}_A = \epsilon^I\{\tilde{p}_A, G_I\}, \\
&\delta_I s^a = \dot{\epsilon}^a\delta_{aI} + \epsilon^I b_I{}^a - s^b\epsilon^I c_{bI}{}^a - v^\beta\epsilon^I c_{\beta I}{}^a, \qquad \delta_I\pi_a = 0, \\
&\delta_I v^\alpha = \dot{\epsilon}^\alpha\delta_{\alpha I}, \qquad \delta_I v^a = (\delta_I s^a)^\cdot
\end{aligned} \tag{26}
$$

leave invariant, modulo to a surface term, the Hamiltonian action (25). Here b, c are structure functions of the gauge algebra (9).

[8]For a theory with first class constraints only, symmetries of extended Hamiltonian action are known [8]. As it was already pointed, our Eq. (25) has a similar form.

[9]Transformation law for v^α turns out to be $\delta v^\alpha = \dot{\epsilon}^\alpha + \epsilon^I b_I{}^\alpha - s^b\epsilon^I c_{bI}{}^\alpha - v^\beta\epsilon^I c_{\beta I}{}^\alpha$, but the last three terms vanish, see end of Section 2.

Thus all the constraints G_I of initial formulation turn out to be infinitesimal generators of the transformations in q^A, p_A-subspace of the phase space.

Symmetries of the reconstructed Lagrangian action. Let us demonstrate that it implies invariance of the Lagrangian action $S_{\tilde{L}} = \int d\tau \tilde{L}$ under the transformations

$$\delta_I q^A = \epsilon^I \left.\{q^A, G_I\}\right|_{p\to\omega(q,\dot{q},s)}, \quad \Leftrightarrow \quad \left\{ \begin{array}{lcl} \delta_I q^\alpha & = & \epsilon^\alpha \delta_{\alpha I}, \\ \delta_I q^i & = & \epsilon^I \left.\frac{\partial G_I}{\partial \tilde{p}_i}\right|_{p\to\omega(q,\dot{q},s)}; \end{array} \right.$$
$$\delta_I s^a = \left.\left(\dot{\epsilon}^a \delta_{aI} + \epsilon^I b_I{}^a - s^b \epsilon^I c_{bI}{}^a - \dot{q}^\beta \epsilon^I c_{\beta I}{}^a\right)\right|_{p\to\omega(q,\dot{q},s)}, \qquad (27)$$

First one notes that variation of $\tilde{L}$ of Eq. (17) under arbitrary transformation $\delta q^A, \delta s^a$ can be presented in the form

$$\delta\tilde{L} = -\dot{\omega}_i \delta q^i - f_\alpha \delta q^\alpha + \dot{q}^\alpha \frac{\partial f_\alpha}{\partial q^A}\delta q^A - \frac{\partial H_0}{\partial q^A}\delta q^A - \delta s^a T_a - s^a \frac{\partial T_a}{\partial q^A}\delta q^A. \qquad (28)$$

We have omitted the term $\left.\frac{\partial \tilde{L}}{\partial \omega_i}\right|_{\omega(q,\dot{q},s)} \delta\omega_i$, the latter is zero as a consequence of the identity (18). To see that $\delta\tilde{L}$ is total derivative, we add the expression $-\epsilon^I \left.\frac{\partial \tilde{L}}{\partial \omega_i}\right|_\omega \left.\frac{\partial G_I}{\partial q^i}\right|_{\tilde{p}\to\omega} \equiv 0$ to r.h.s. of Eq. (28). With δq^A given by Eq. (27), one obtains after some algebra

$$\delta\tilde{L} = \left[-\dot{T}_a\epsilon^a + \dot{q}^\alpha\epsilon^\beta\left(\frac{\partial f_\alpha}{\partial q^\beta} - \frac{\partial f_\beta}{\partial q^\alpha} - \left(\frac{\partial f_\alpha}{\partial q^i}\frac{\partial f_\beta}{\partial \tilde{p}_i} - \alpha\leftrightarrow\beta\right)\right) - (s^a\epsilon^\alpha - \dot{q}^\alpha\epsilon^a)\left(\frac{\partial T_a}{\partial q^\alpha} + \left(\frac{\partial f_\alpha}{\partial q^i}\frac{\partial T_a}{\partial \tilde{p}_i} - f\leftrightarrow T\right)\right) - \epsilon^\alpha\left(\frac{\partial H_0}{\partial q^\alpha} - \left(\frac{\partial H_0}{\partial q^i}\frac{\partial f_\alpha}{\partial \tilde{p}_i} - H_0\leftrightarrow f\right)\right) - \epsilon^a\left(\frac{\partial H_0}{\partial q^i}\frac{\partial T_a}{\partial \tilde{p}_i} - H_0\leftrightarrow T\right) - s^a\epsilon^b\left(\frac{\partial T_a}{\partial q^i}\frac{\partial T_b}{\partial \tilde{p}_i} - a\leftrightarrow b\right) - \delta s^a T_a\right]\Bigg|_{p\to\omega}$$
$$\equiv \left[\dot{\epsilon}^a T_a - \dot{q}^\alpha\epsilon^\beta\{\Phi_\alpha,\Phi_\beta\} + (s^a\epsilon^\alpha - \dot{q}^\alpha\epsilon^a)\{\Phi_\alpha, T_a\} - \epsilon^\alpha\{H_0,\Phi_\alpha\} - \epsilon^a\{H_0, T_a\} - s^a\epsilon^b\{T_a,T_b\} - \delta s^a T_a\right]\Big|_{p\to\omega},$$
$$= \left[\left(\dot{\epsilon}^a - \dot{q}^\alpha\epsilon^I c_{\alpha I}{}^a + \epsilon^I b_I{}^a - s^b\epsilon^I c_{bI}{}^a - \delta s^a\right)T_a\right]\Big|_{\tilde{p}\to\omega}. \qquad (29)$$

Then the variation of s^a given by Eq. (27) implies $\delta\tilde{L} = div$, as it has been stated.

Symmetries of the initial action. Let us consider a combination of the symmetries (27) $\delta \equiv \sum_I \delta_I$, which obeys: $\delta s^a = 0$, for all s^a. The Lagrangian $\tilde{L}(q,s)$ will be invariant under this symmetry for any fixed value of s^a, in particular, for $s^a = 0$. But owing to Eqs. (19), (15), (3), the reconstructed Lagrangian coincides with the initial one for $s^a = 0$: $\tilde{L}(q,0) = L(q)$. So the initial action will be invariant under any transformation

$$\delta q^A = \sum_I \delta_I q^A \bigg|_{s=0}, \tag{30}$$

which obeys to the system $\delta s^a|_{s=0}$, that is

$$\dot{\epsilon}^a + \epsilon^I b_I{}^a - v^\beta \epsilon^I c_{\beta I}{}^a = 0, \qquad a = 1,2,\ldots,[a]. \tag{31}$$

One has $[a]$ equations for $[\alpha] + [a]$ variables ϵ^I. Similarly to Ref. [2], the equations can be solved by pure algebraic methods, which give some $[a]$ of ϵ in terms of the remaining ϵ and their derivatives of order less than N. It allows one to find $[\alpha]$ local symmetries of L. As it was already mentioned, the problem here is to prove the completeness and the irreducibility of the set.

As an illustration, we look for local symmetries of a theory with fourth-stage constraints presented in the initial formulation. Let us consider the Lagrangian

$$L = \frac{1}{2}(\dot{x})^2 + \xi(x)^2, \tag{32}$$

where $x^\mu(\tau), \xi(\tau)$ are configuration space variables, $\mu = 0,1,\ldots,n$, $(x)^2 \equiv \eta_{\mu\nu}x^\mu x^\nu$, $\eta_{\mu\nu} = (-,+,\ldots,+)$. One notes that the theory is manifestly invariant under global transformations of $SO(1,n-1)$-group. As it will be seen, our procedure preserves the invariance.

Denoting the conjugate momenta for x^μ, ξ as p_μ, p_ξ, one obtains the complete Hamiltonian

$$H_0 = \frac{1}{2}p^2 - \xi(x)^2 + v_\xi p_\xi, \tag{33}$$

where v_ξ is multiplier for the primary constraint $p_\xi = 0$. The complete system of constraints turns out to be

$$\Phi_1 \equiv p_\xi = 0, \quad T_2 \equiv x^2 = 0, \quad T_3 \equiv xp = 0, \quad T_4 \equiv p^2 = 0. \tag{34}$$

For the case, the variable ξ plays the role of q^α, while x^μ play the role of q^i. The constraints obey to the gauge algebra (9), with non vanishing coefficient functions being

$$\begin{array}{l} c_{23}{}^2 = -c_{32}{}^2 = 2, \qquad c_{24}{}^3 = -c_{42}{}^3 = 4, \qquad c_{34}{}^4 = -c_{43}{}^4 = 2; \\ b_1{}^2 = 1, \qquad b_2{}^3 = 2, \qquad b_3{}^4 = 1, \qquad b_3{}^3 = 2\xi, \qquad b_4{}^3 = 4\xi. \end{array} \quad (35)$$

Equations (10), (13) acquire the form

$$\tilde{H} = \frac{1}{2}(1+2s^4)\tilde{p}^2 - \xi x^2 + s^2(x)^2 + s^3(xp) + v_\xi p_\xi + v^a\pi_a, \quad (36)$$

$$\tilde{p}^\mu = \frac{\dot{x}^\mu - s^3 x^\mu}{1+2s^4}. \quad (37)$$

Using these equations, one writes the reconstructed Lagrangian (17)

$$\tilde{L} = \frac{1}{2(1+2s^4)}(x^\mu - s^3 x^\mu)^2 + (\xi - s^2)(x^\mu)^2. \quad (38)$$

It suggests the following redefinition of variables: $1+2s^3 \equiv e, \xi - s^2 \equiv \xi_1$, then the previous expression can be written in the form

$$\tilde{L}(e, \xi_1) = \frac{1}{2e}(x^\mu - s^3 x^\mu)^2 + \xi_1 (x^\mu)^2. \quad (39)$$

Note that the reconstructed Lagrangians (38), (39) remain invariant under $SO(1, n-1)$ global transformations.

The Lagrangian (38) implies four primary constraints $p_\xi = 0, \pi_a = 0$, and four secondary constraints (34). The corresponding complete Hamiltonian is given by Eq. (36). It has four irreducible local symmetries, the corresponding parameters are denoted as $\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4$. By using of Eqs. (26), (35), manifest form of the symmetries can be written immediately as follows (we have omitted the variations $\delta_i v^a = (\delta_i s^a)^\cdot$)

$$\delta_1 \xi = \epsilon^1, \quad \delta_1 s^2 = \epsilon^1, \quad \delta_1 v_\xi = \dot{\epsilon}^1, \quad (40)$$

$$\delta_2 \tilde{p}^\mu = -2\epsilon^2 x^\mu, \quad \delta_2 s^2 = \dot{\epsilon}^2 + 2\epsilon^2 s^3, \quad \delta_2 s^3 = 2\epsilon^2(1+2s^4); \quad (41)$$

$$\begin{array}{l} \delta_3 x^\mu = \epsilon^3 x^\mu, \quad \delta_3 \tilde{p}^\mu = -\epsilon^3 \tilde{p}^\mu, \\ \delta_3 s^2 = 2\epsilon^3(\xi - s^2), \quad \delta_3 s^3 = \dot{\epsilon}^3, \quad \delta_3 s^4 = \epsilon^3(1+2s^4); \end{array} \quad (42)$$

$$\delta_4 x^\mu = 2\epsilon^4\tilde{p}^\mu, \quad \delta_4 s^3 = 4\epsilon^4(\xi - s^2), \quad \delta_4 s^4 = \dot{\epsilon}^4 - 2\epsilon^4 s^3. \tag{43}$$

The corresponding symmetries for $\tilde{L}$ are obtained according to Eq. (27)

$$\delta_1\xi = \epsilon^1, \qquad \delta_1 s^2 = \epsilon^1; \tag{44}$$

$$\delta_2 s^2 = \dot{\epsilon}^2 + 2\epsilon^2 s^3, \qquad \delta_2 s^3 = 2\epsilon^2(1 + 2s^4); \tag{45}$$

$$\delta_3 x^\mu = \epsilon^3 x^\mu, \ \delta_3 s^2 = 2\epsilon^3(\xi - s^2), \ \delta_3 s^3 = \dot{\epsilon}^3, \ \delta_3 s^4 = \epsilon^3(1 + 2s^4); \tag{46}$$

$$\delta_4 x^\mu = 2\epsilon^4\frac{\dot{x}^\mu - s^3 x^\mu}{1 + 2s^4}, \quad \delta_4 s^3 = 4\epsilon^4(\xi - s^2), \quad \delta_4 s^4 = \dot{\epsilon}^4 - 2\epsilon^4 s^3. \tag{47}$$

From these expressions one can write also the symmetries for $L(e, \xi_1)$ of Eq. (39). The symmetry (44) disappears, since $L(e, \xi_1)$ is constructed from it's gauge invariant variables. The remaining symmetries acquire the form

$$\delta_2\xi_1 = -\dot{\epsilon}^2 - 2\epsilon^2 s^3, \qquad \delta_2 s^3 = 2e\epsilon^2; \tag{48}$$

$$\delta_3 x^\mu = \epsilon^3 x^\mu, \ \delta_3\xi_1 = -2\epsilon^3\xi_1, \ \delta_3 s^3 = \dot{\epsilon}^3, \ \delta_3 e = 2\epsilon^3 e; \tag{49}$$

$$\delta_4 x^\mu = \frac{2\epsilon^4}{e}(\dot{x}^\mu - s^3 x^\mu), \quad \delta_4 s^3 = 4\epsilon^4\xi_1, \quad \delta_4 s^4 = \dot{\epsilon}^4 - 2\epsilon^4 s^3. \tag{50}$$

δ_4-symmetry can be replaced by the combination $\delta_\epsilon \equiv \delta(\epsilon^4 = \frac{1}{2}\epsilon e) + \delta(\epsilon^3 = \epsilon s^3) + \delta(\epsilon^2 = -\epsilon\xi_1)$, the latter has more simple form as compare with (50)

$$\delta_\epsilon x^\mu = \epsilon\dot{x}^\mu, \quad \delta_\epsilon\xi_1 = (\epsilon\xi_1)^{\cdot}, \quad \delta_\epsilon s^3 = (\epsilon s^3)^{\cdot}, \quad \delta_\epsilon e = (\epsilon e)^{\cdot}. \tag{51}$$

As an independent symmetries of $L(e, \xi_1)$, one can take either Eqs. (48)-(50), or Eqs. (48), (49), (51).

Since the initial Lagrangian L implies unique chain of four first class constraints, one expects one local symmetry of $\overset{(3)}{\epsilon}$-type [4]. It can be find according to defining equations (31), for the case

$$\begin{array}{lllll} \epsilon^1 & +\dot{\epsilon}^2 & +2\epsilon^3\xi & & = 0, \\ & 2\epsilon^2 & +\dot{\epsilon}^3 & +4\epsilon^4\xi & = 0, \\ & & \epsilon^3 & +\dot{\epsilon}^4 & = 0. \end{array} \tag{52}$$

It allows one to find $\epsilon^1, \epsilon^2, \epsilon^3$ in terms of $\epsilon^4 \equiv \epsilon$: $\epsilon^1 = -\frac{1}{2}\overset{(3)}{\epsilon} + 4\dot{\epsilon}\xi + 2\epsilon\dot{\xi}$, $\epsilon^2 = \frac{1}{2}\ddot{\epsilon} - 2\epsilon\xi$, $\epsilon^3 = -\dot{\epsilon}$. Then Eq. (30) gives the local symmetry of the Lagrangian (32)

$$\delta x^\mu = -\dot{\epsilon} x^\mu + 2\epsilon \dot{x}^\mu, \qquad \delta\xi = -\frac{1}{2}\overset{(3)}{\epsilon} + 4\dot{\epsilon}\xi + 2\epsilon\dot{\xi}. \tag{53}$$

In resume, in this work we have presented a relatively simple way for finding the complete irreducible set of local symmetries in a Lagrangian theory with first class constraints. Instead of looking for the symmetries of initial Lagrangian L, one can construct an equivalent Lagrangian $\tilde{L}$ given by Eq. (17), the latter implies at most secondary first class constraints. Local symmetries of $\tilde{L}$ can be immediately written according to Eq. (27).

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